Calcul intégral — 2ème Bac Pro

Correction des exercices du polycopié du cours

EX.1 Vérifier les calculs suivants :

1°/ $\int_{-1}^{0}(2x+1)\,dx = 0$

2°/ $\int_0^1 (x^2+6x+1)\,dx = \frac{13}{3}$

3°/ $\int_4^{16}\frac{dx}{2\sqrt{x}} = 2$; 4°/ $\int_0^3 \frac{dx}{x+2} = \ln\left(\frac{5}{2}\right)$

solution :

1°/ $\int_{-1}^{0}(2x+1)\,dx \overset{?}{=} 0$

on a : $\int_{-1}^{0}(2x+1)\,dx = \int_{-1}^{0} f(x)\,dx$

avec : $f(x) = 2x+1$

une primitive de f : $F(x) = x^2+x$

car : $F'(x) = (x^2+x)' = 2x+1$

donc : $\int_{-1}^{0}(2x+1)\,dx = \int_{-1}^{0} F'(x)\,dx$

$= [F(x)]_{-1}^{0} = [x^2+x]_{-1}^{0}$

$= 0^2+0-((-1)^2+(-1)) = -(1-1) = \boxed{0}$

2°/ $\int_0^1 \underbrace{(x^2+6x+1)}_{f(x)}\,dx \overset{?}{=} \frac{13}{4}$

$\Rightarrow F(x) = \frac{x^3}{3} + 6\frac{x^2}{2} + x$, F est une primitive.

donc :

$\int_0^1 f(x)\,dx = [F(x)]_0^1 = F(1) - F(0)$

avec : $\begin{cases} F(1) = \frac{1}{3} + \frac{6}{2} + 1 = \frac{1}{3} + 4 = \frac{13}{3} \\ F(0) = 0+0+0 = 0 \end{cases}$

donc : $\int_0^1 (x^2+6x+1)\,dx = \boxed{\frac{13}{3}}$

3°/ $\int_4^{16}\frac{dx}{2\sqrt{x}} = \int_4^{16}\underbrace{\frac{1}{2\sqrt{x}}}_{f(x)}\,dx \overset{?}{=} 2$

$f(x) = \frac{1}{2\sqrt{x}} \Rightarrow F(x) = \sqrt{x}$

donc : $\int_4^{16} f(x)\,dx = [\sqrt{x}]_4^{16} = \sqrt{16} - \sqrt{4}$

$= 4 - 2 = \boxed{2}$

4°/ $\int_0^3 \frac{dx}{x+2} = \int_0^3 \frac{1}{x+2}\,dx \overset{?}{=} \ln\left(\frac{5}{2}\right)$

$f(x) = \frac{1}{x+2} = \frac{(x+2)'}{x+2} = \ln'(x+2)$

$F(x) = \ln(x+2)$ donc :

$\int_0^3 \frac{dx}{x+2} = F(x)]_0^3 = [\ln(x+2)]_0^3$

$= \ln(3+2) - \ln(0+2) = \ln 5 - \ln 2$

$= \boxed{\ln\left(\frac{5}{2}\right)}$

EX.2 Montrer que :

1°/ $\int_0^{\pi}\left(4x + \frac{2}{3}\sin(x)\right)dx = 2\pi^2 + \frac{4}{3}$

2°/ $\int_0^1 (5x^3 + e^x)\,dx = e + \frac{1}{4}$

3°/ $\int_1^e \left(\frac{1}{x} - \frac{1}{x^2}\right)dx = \frac{1}{e}$

4°/ $\int_1^2 (2x+3)(x^2+3x)^2\,dx = 312$

Solution :

1°/ $\int_0^{\pi}\left(4x + \frac{2}{3}\sin(x)\right)dx$

$= 4\times\int_0^{\pi} x\,dx + \frac{2}{3}\int_0^{\pi}\sin(x)\,dx$

$= 4\left[\frac{x^2}{2}\right]_0^{\pi} + \frac{2}{3}[-\cos(x)]_0^{\pi}$

$= 4\left[\frac{\pi^2}{2} - 0\right] + \frac{2}{3}[-(-1)+1]$

$= 4\times\frac{\pi^2}{2} + \frac{2}{3}\times 2 = \boxed{2\pi^2 + \frac{4}{3}}$

2°/ $\int_0^1 (5x^3+e^x)\,dx = 5\times\int_0^1 x^3\,dx + \int_0^1 e^x\,dx$

$= 5\left[\frac{x^4}{4}\right]_0^1 + [e^x]_0^1 = 5\times\left(\frac{1}{4} - 0\right) + e^1 - e^0$

$$\int_0^1 (5x^3+e^x)dx = \frac{5}{4} + e - 1$$
$$= e + \frac{5}{4} - 1 = e + \frac{1}{4}$$

3°/ $$\int_1^e \frac{1}{x} - \frac{1}{x^2}\, dx = \int_1^e \frac{1}{x} dx + \int_1^e -\frac{1}{x^2} dx$$
$$= \left[\ln(x)\right]_1^e + \left[\frac{1}{x}\right]_1^e$$
$$= \ln(e) - \ln(1) + \frac{1}{e} - \frac{1}{1}$$
$$= 1 - 0 + \frac{1}{e} - 1 = \boxed{\frac{1}{e}}$$

car : $\ln'(x) = \frac{1}{x}$ et $\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$

donc : $\frac{1}{x} - \frac{1}{x^2} = \left(\ln(x) + \frac{1}{x}\right)'$

4°/ $$\int_1^2 \underbrace{(2x+3)\,(x^2+3x)^2}_{f(x)}\, dx$$
$$f(x) = (x^2+3x)'(x^2+3x)^2$$
$$= \frac{1}{3} \times \left[3\,(x^2+3x)'(x^2+3x)^2\right]$$
$$= \frac{1}{3} F'(x) \;;\; \text{avec : } F(x) = (x^2+3x)^3$$

donc : $\int_1^2 f(x)dx = \frac{1}{3}\left[F(x)\right]_1^2 = \frac{1}{3}(F(2)-F(1))$

$$\begin{cases} F(2) = (2^2+3\times 2)^3 = (4+6)^3 = 1000 \\ F(1) = (1+3)^3 = 4^3 = 64 \end{cases}$$

$$\int_1^2 f(x)dx = \frac{1}{3}\left[1000 - 64\right] = \frac{936}{3} = \boxed{312}$$

EX.3 Montrer que :

1°/ $\int_0^3 2x\, e^{x^2} dx = e^9 - 1$

2°/ $\int_1^2 \sqrt{x}\, dx = \frac{4\sqrt{2}}{3} - \frac{2}{3}$

3°/ $\int_1^2 \frac{2x+3}{x^2+3x} dx = \ln\left(\frac{5}{2}\right)$

Solution : 1°/ $\int_0^3 2x e^{x^2} dx$

$$f(x) = (x^2)' e^{x^2} = \left(e^{x^2}\right)' = F'(x)$$

avec : $F(x) = e^{x^2}$ donc :

$$\int_0^3 f(x)dx = \left[F(x)\right]_0^3 = F(3) - F(0)$$
$$= e^{3^2} - e^{0^2} = e^9 - e^0 = \boxed{e^9 - 1}$$

2°/ $$\int_1^2 \sqrt{x}\, dx = \int_1^2 x^{1/2}\, dx$$
$$= \left[\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_1^2 = \left[\frac{x^{\frac{1}{2}+1}}{3/2}\right]_1^2$$
$$= \frac{2^{\frac{1}{2}+1}}{3/2} - \frac{1}{3/2} = \frac{2^{1/2} \times 2^1}{3/2} - \frac{2}{3}$$
$$= \sqrt{2} \times 2 \times \frac{2}{3} - \frac{2}{3} = \boxed{\frac{4}{3}\sqrt{2} - \frac{2}{3}}$$

3°/ $$\int_1^2 \frac{2x+3}{x^2+3x} dx = \int_1^2 \frac{(x^2+3x)'}{x^2+3x} dx$$
$$= \int_1^2 \ln'(x^2+3x)\, dx = \left[\ln(x^2+3x)\right]_1^2$$
$$= \ln(2^2+3\times 2) - \ln(1+3\times 1)$$
$$= \ln(10) - \ln(4) = \ln\left(\frac{10}{4}\right) = \boxed{\ln\left(\frac{5}{2}\right)}$$

EX.4 A l'aide d'une intégration par partie montrer que :

1°/ $\int_{-1}^2 x e^x\, dx = e^2 + \frac{2}{e}$

2°/ $\int_0^{\pi} x \sin(x)\, dx = \pi$

3°/ $\int_2^e 4x^3 \ln(x)\, dx = 3\frac{e^4}{4} - 16\ln(2) + 4$

Solution :

1°/ $$\int_{-1}^2 x e^x dx = \int_{-1}^2 u(x)\, v'(x)\, dx$$

avec : $\begin{cases} u(x) = x \\ v'(x) = e^x \end{cases}$ donc : $\begin{cases} u'(x) = 1 \\ v(x) = e^x \end{cases}$

par Suite : $\int u(x)v'(x)dx = [u(x)v(x)] - \int u'(x)v(x)dx$

$\Rightarrow \int_{-1}^{2} xe^x dx = [xe^x]_{-1}^{2} - \int_{-1}^{2} 1 \times e^x dx$

$= 2e^2 + e^{-1} - \int_{-1}^{2} e^x dx$ $\quad \left\{ e^{-1} = \frac{1}{e} \right.$

$= 2e^2 + e^{-1} - [e^x]_{-1}^{2} = 2e^2 + e^{-1} - (e^2 - e^{-1})$

$= 2e^2 + e^{-1} - e^2 + e^{-1} = \boxed{\boxed{e^2} + \frac{2}{e}}$

2°/ $\int_0^{\pi} x\sin(x)dx = \int_0^{\pi} \underbrace{x}_{u(x)} \times \underbrace{\sin(x)}_{v'(x)} dx$

$\begin{cases} u(x) = x \\ v'(x) = \sin(x) \end{cases} \Rightarrow \begin{cases} u'(x) = 1 \\ v(x) = -\cos(x) \end{cases}$

donc

$\int u(x)v'(x)dx = [u(x)v(x)] - \int u'(x)v(x)dx$

$\int_0^{\pi} x\sin(x)dx = [-x\cos(x)]_0^{\pi} - \int_0^{\pi} 1 \times (-\cos(x))dx$

$= [-\pi(-1) + 0] + \int_0^{\pi} \cos(x)dx$

$= \pi + [\sin(x)]_0^{\pi} = \pi + (0 - 0) = \boxed{\pi}$

3°/ $\int_2^{e} 4x^3\ln(x)dx = \int_2^{e} (4x^3)\ln(x)dx$

$\begin{cases} u'(x) = 4x^3 \\ v(x) = \ln(x) \end{cases} \Rightarrow \begin{cases} u(x) = x^4 \\ v'(x) = \frac{1}{x} \end{cases}$

$\int_2^{e} 4x^3\ln(x)dx = [u(x)v(x)]_2^{e} - \int_2^{e} u(x)v'(x)dx$

$= [x^4\ln(x)]_2^{e} - \int_2^{e} x^4 \times \frac{1}{x} dx$

$= e^4\ln(e) - 2^4\ln(2) - \int_2^{e} x^3 dx$

$= e^4 \times 1 - 16\ln(2) - \left[\frac{x^4}{4}\right]_2^{e}$

$= e^4 - 16\ln(2) - \left(\frac{e^4}{4} - \frac{2^4}{4}\right)$

$= e^4 - \frac{e^4}{4} - 16\ln(2) + \frac{16}{4}$

$= \left(1 - \frac{1}{4}\right)e^4 - 16\ln(2) + 4$

$= \boxed{\frac{3}{4}e^4 - 16\ln(2) + 4}$